Lecture 4.

Tricks $\begin{cases} ① \text{ Correlations} \quad Nu = fn(Re, Pr) \\ ② \text{ Similarity (momentum \& energy)} \\ ③ \text{ Integral method.} \end{cases}$

Integral Method

Assuming we don't know the exact solution of $T(y)$, within δ_t, but we approximate the solution using a polynomial function:

$$T(y) = A + By + Cy^2 + Dy^3 . \quad \cdots ①$$

We want to get the coefficients A, B, C, D.

① is not convenient, as it has dimensions.
Non-dimensionalize:

(1-1) $\rightarrow$ non-dimensional temperature

$$\boxed{\frac{T - T_w}{T_\infty - T_w} = a + b\left(\frac{y}{\delta_t}\right) + c\left(\frac{y}{\delta_t}\right)^2 + d\left(\frac{y}{\delta_t}\right)^3} \quad \cdots ②$$

boundary conditions:
(B.C.)

$$\begin{cases} \text{at } \dfrac{y}{\delta_t} = 0, & \dfrac{T - T_w}{T_\infty - T_w} = 0 \\ \text{at } \dfrac{y}{\delta_t} = 1, & \dfrac{T - T_w}{T_\infty - T_w} = 1 \\ \text{at } \dfrac{y}{\delta_t} = 1, & \dfrac{\partial\left(\frac{T - T_w}{T_\infty - T_w}\right)}{\partial\left(\frac{y}{\delta_t}\right)} = 0 \\ \text{at } \dfrac{y}{\delta_t} = 0, & \dfrac{\partial^2\left(\frac{T - T_w}{T_\infty - T_w}\right)}{\partial\left(\frac{y}{\delta_t}\right)^2} = 0 . \end{cases}$$

B.C. ①-③ are more obvious.
B.C. ④ comes from energy equation at the wall:

$$\underbrace{u\cancel{\frac{\partial T}{\partial x}}}_{0} + \underbrace{v\cancel{\frac{\partial T}{\partial y}}}_{0} = \alpha \frac{\partial^2 T}{\partial y^2}$$

both 0 since no slip at the wall

solving the coefficients:

B.C. ① $\Rightarrow$ $a = 0$

B.C. ② $\Rightarrow$ $1 = 0 + b + c + d$

B.C. ③ $\Rightarrow$ $0 = b + 2c + 3d$

B.C. ④ $\Rightarrow$ $0 = 2c$

We obtain: $d = -\frac{1}{2}$, $b = \frac{3}{2}$, so

$$\boxed{\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\left(\frac{y}{\delta_t}\right) - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3} \quad \cdots \text{③}$$

Comment: if you define a temperature of $\frac{T - T_\infty}{T_w - T_\infty}$

you will get $$\boxed{\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{3}{2}\left(\frac{y}{\delta_t}\right) + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3} \quad \text{③}^*$$

Equation ③ looks simple, yet δ_t is still unknown.

heat flux $\quad q = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} = -k \left.\frac{\partial\left(\frac{T-T_w}{T_\infty - T_w}\right)}{\partial\left(\frac{y}{\delta_t}\right)}\right|_{\frac{y}{\delta_t}=0} \cdot \frac{T_\infty - T_w}{\delta_t}$

Use ③, we get : $q = -k \frac{T_\infty - T_w}{\delta_t} \cdot \frac{3}{2} = k \frac{T_w - T_\infty}{\delta_t} \frac{3}{2}$

$$\Rightarrow \boxed{h = \frac{q}{T_w - T_\infty} = \frac{3k}{2\delta_t} = \frac{3}{2} \cdot \frac{k}{\delta} \cdot \frac{\delta}{\delta_t}} \quad \cdots ④.$$

What are $\frac{k}{\delta}$ and $\frac{\delta}{\delta_t}$?

$\boxed{\text{Integral Method}}$

Integrate boundary layer energy equation from wall ($y=0$) to $y=\delta_t$.

$$\int_0^{\delta_t} u \frac{\partial T}{\partial x} dy + \int_0^{\delta_t} v \frac{\partial T}{\partial y} dy = \alpha \int_0^{\delta_t} \frac{\partial^2 T}{\partial y^2} dy$$

$$\underbrace{\int_0^{\delta_t} \frac{\partial (uT)}{\partial x} dy - \int_0^{\delta_t} T \frac{\partial u}{\partial x} dy} + \underbrace{\int_0^{\delta_t} \frac{\partial (vT)}{\partial y} dy - \int_0^{\delta_t} T \frac{\partial v}{\partial y} dy} = \alpha \left.\frac{\partial T}{\partial y}\right|_{y=0}^{y=\delta_t}$$

$$\int_0^{\delta_t} \frac{\partial (uT)}{\partial x} dy + \underbrace{\left. vT \right|_0^{\delta_t}}_{=T_\infty V_\infty|_{y=\delta_t}} - \int_0^{\delta_t} T \underbrace{\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}}_{=0,\ \text{continuity equation}} dy = \alpha \left[0 - \left.\frac{\partial T}{\partial y}\right|_{y=0} \right]$$

What is $V_\infty|_{y=\delta_t}$? integrate continuity equation : $\int_0^{\delta_t} \frac{\partial u}{\partial x} dy + \int_0^{\delta_t} \frac{\partial v}{\partial y} dy = 0$

$$v|_{y=\delta_t} = -\int_0^{\delta_t} \frac{\partial u}{\partial x} dy$$

So, back to energy equation (integrated):

$$\int_0^{\delta t} \frac{\partial (uT)}{\partial x} dy + \left[-\int_0^{\delta t} T_\infty \frac{\partial u}{\partial x} dy \right] = -\alpha \frac{\partial T}{\partial y}\bigg|_{y=0} = \frac{q_w}{\rho c_p} \qquad \boxed{\alpha = \frac{k}{\rho c_p}}$$

$$\boxed{\int_0^{\delta t} \frac{\partial [u \cdot (T - T_\infty)]}{\partial x} dy = -\alpha \frac{\partial T}{\partial y}\bigg|_{y=0}}$$

or,
$$\boxed{\frac{d}{dx} \int_0^{\delta t} u (T - T_\infty) dy = -\alpha \frac{\partial T}{\partial y}\bigg|_{y=0} = \frac{q_w}{\rho c_p}} \quad \cdots (5)$$

Now, substitute (3)* into (5):

note we also need u, but similarly, $\boxed{\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3}$

$$u_\infty \cancel{(-T_\infty + T_w)} \frac{d}{dx}\left[\delta_t \int_0^1 \frac{u}{u_\infty} \underline{\left(\frac{T - T_\infty}{T_w - T_\infty}\right)} d\left(\frac{y}{\delta_t}\right)\right] = -\alpha \frac{\cancel{T_w - T_\infty}}{\delta_t} \frac{d\left(\frac{T - T_\infty}{T_w - T_\infty}\right)}{d(y/\delta_t)}\Bigg|_{\frac{y}{\delta_t}=0}$$

assume $\delta_t \le \delta$ ↑ use equation (3)*

$$\delta_t \frac{d}{dx}\left[\delta_t \underbrace{\int_0^1 \left(\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)\left(1 - \frac{3}{2}\left(\frac{y}{\delta_t}\right) + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right) d\left(\frac{y}{\delta_t}\right)}_{= \frac{3}{20}\left(\frac{\delta_t}{\delta}\right) - \frac{3}{280}\left(\frac{\delta_t}{\delta}\right)^3}\right] = \frac{3}{2} \frac{\alpha}{u_\infty}$$

let $\frac{\delta_t}{\delta} = \phi = \text{fn}(Pr)$ constant

separate variables: $2 \delta_t \frac{d\delta_t}{dx} = \frac{d \delta_t^2}{dx} = \dfrac{3\alpha / u_\infty}{\frac{3}{20}\phi - \frac{3}{280}\phi^3}$

Integrate again: $\boxed{\delta_t = \sqrt{\frac{3\alpha x}{u_\infty}} \Big/ \sqrt{\frac{3}{20}\phi - \frac{3}{280}\phi^3}}$ (6)

Also from integral method of momentum equation,

we can get : $\boxed{\delta = \frac{4.64x}{\sqrt{Re_x}}}$ ⑦ ← this is close to exact solution $\delta = \frac{4.92x}{\sqrt{Re_x}}$

divide ⑥ by ⑦, ⇒

$$\frac{\delta_t}{\delta} = \phi = \frac{0.9638}{\sqrt{Pr\,\phi\,(1-\phi^2/14)}}$$

⑧

rearranging : $\boxed{\frac{\delta_t}{\delta} = \frac{1}{1.025\,Pr^{1/3}\left[1-\left(\delta_t^2/14\delta^2\right)\right]^{1/3}} \approx \frac{1}{1.025\,Pr^{1/3}}}$

exact solution is $\boxed{\frac{\delta_t}{\delta} = Pr^{-1/3}}$ ⑧* for $0.6 \le Pr \le 50$

⑦ and ⑧ into ④, we finally have

$$h = \frac{3}{2} \cdot \frac{k}{\delta} \cdot \frac{\delta}{\delta_t} \Rightarrow Nu_x = \frac{hx}{k} = \frac{3}{2}\frac{\sqrt{Re_x}}{4.64} \times 1.025\,Pr^{1/3}$$

$$= 0.3314\,Re_x^{1/2}\,Pr^{1/3}$$